Modern Portfolio Theory

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Risk and Variation

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Return Notation

notation	description	formula	example
r^i	return rate of asset i		
r^f	risk-free return rate		
$\tilde{r}^i$	excess return rate of asset i	$r^i - r^f$	

Return Notation Continued

Table: Portfolio example

	return	allocation weight
bonds	r^b	w
stocks	r^s	$1 - w$

Table: Return statistics notation

mean	variance	correlation
μ	σ^2	ρ

Where weight allocation is the percentage of each holding comprises in an investment portfolio

Portfolio Return Stats

$$\mu^p = w\mu^b + (1-w)\mu^s$$

$$\sigma_p^2 = w^2\sigma_b^2 + (1-w)^2\sigma_s^2 + 2w(1-w)\rho\sigma_s\sigma_b$$

Imperfect Correlation

- The volatility function is convex

$$\sigma_\rho < w\sigma_b + (1-w)\sigma_s$$

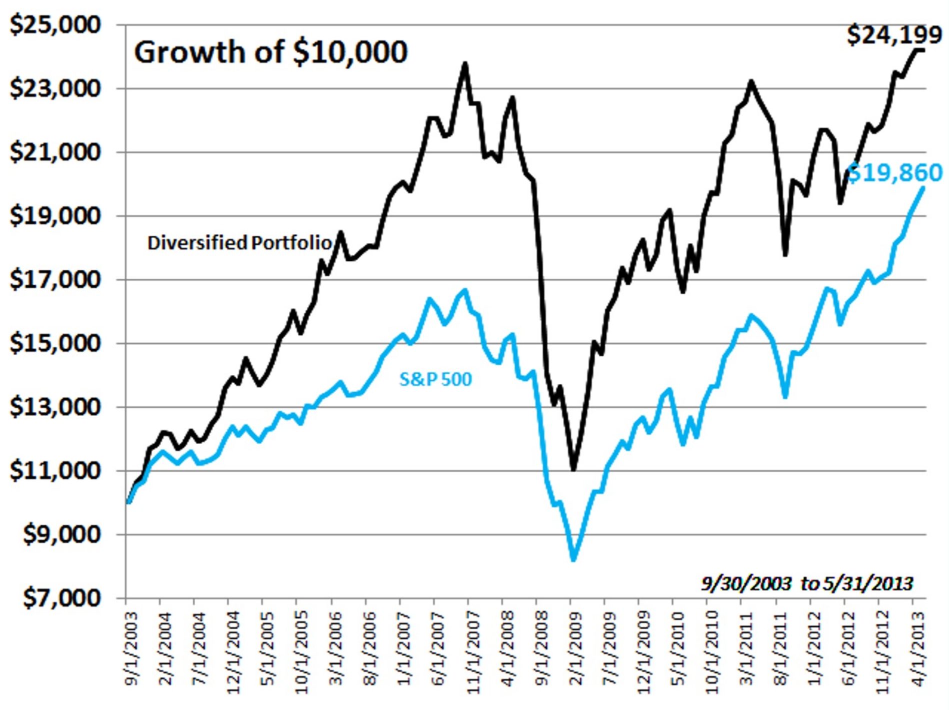
- Yet the mean return is still linear in the portfolio allocation

$$\mu^\rho = w\mu^b + (1-w)\mu^s$$

Diversification

Portfolio diversification refers to the case where:

- Mean returns are linear in allocations
- While volatility of returns is less than linear in a
- However, this is only required when p < 1
    - Opportunity to diversify portfolio risk away



Portfolio Variance as Average Covariances

- Suppose that asset returns have
  - Identical volatilities
  - Identical correlations

$$Correlation = \frac{Cov\,(x, y)}{\sigma x * \sigma y}$$

- Using the following notation for averaging variances and covariances across the n assets

$$\overline{\sigma_i}^2 \equiv \frac{1}{n} \sum_{i=1}^{n} \sigma_i^2$$

$$\overline{\sigma_{i,j}} \equiv \frac{1}{n(n-1)} \sum_{j \neq i} \sum_{i=1}^{n} \sigma_{i,j}$$

Portfolio Variance Decomposition

If we take an equally weighted portfolio, $w_i = 1/n$.

$$\sigma_p^2 = \frac{1}{n}\overline{\sigma_i}^2 + \frac{n-1}{n}\overline{\sigma_{i,j}}$$

We now note that variance has a term that can be diversified to zero, and another term that remains

Systematic Risk

$$\sigma_p^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2$$

$$\lim_{n\to\infty} \sigma_p^2 \to \underbrace{\rho\sigma^2}_{\text{systematic}}$$

- A fraction, p, of the variance is systematic
- No amount of diversification can get the portfolio variance lower

$$\sigma_p^2 \geq \rho\sigma^2$$

1. The inequality at the bottom holds for any n and any set of allocation weights w i

Idiosyncratic Risk

$$\sigma_p^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2$$

- Idiosyncratic risk refers to the diversifiable part of $\sigma_p^{\,2}$
- An equally weighted portfolio has idiosyncratic risk equal to $\sigma^2/2$
- For general weights, w^i, remaining idiosyncratic is risk bounded my $\max_i w^i\sigma^2$

Correlation and Diversified Portfolios

- For p =1, there is no possible diversification, regardless of n.

$$\sigma_\rho^2 = \sigma^2$$

- For p = 0, there is no systematic risk, only variance or idiosyncratic risk is remaining

$$\sigma_\rho^2 = \frac{1}{n}\sigma^2$$

- In this case, as n get large, the portfolio can theoretically become riskless

$$\lim_{n\to\infty} \sigma_\rho^2 = 0$$

Portfolio Irrelevance of Individual Security Variance

- As the number of securities in portfolio, n, gets large,

$$\lim_{n \to \infty} \sigma_p^2 = \overline{\sigma_{i,j}}$$

- Individual security variance is unimportant and therefore, overall portfolio variance is the average of individual security covariance

Mean Variance

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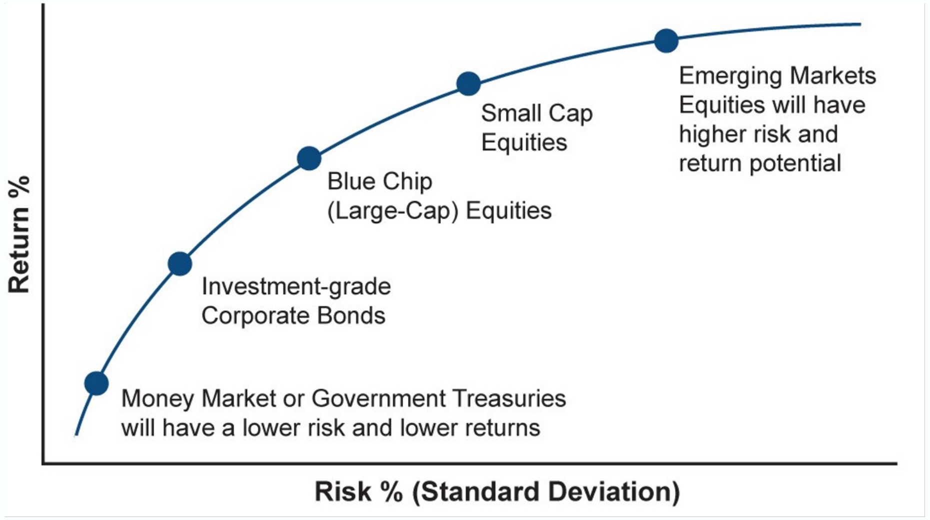
Mean-Variance Portfolio Allocation

- We want to create a portfolio that given any N assets, it optimizes the risk to return profile of that portfolio
  - Mean excess returns as a measurement of portfolio benefit
  - Average variance to measure risk

$$\text{Sharpe Ratio} = \frac{\mu^p - r^f}{\sigma^p} = \frac{\tilde{\mu}^p}{\sigma^p}$$



Risk and Return Tradeoff

- Traditional risk and return analysis states that higher return must always equal higher risk
- According to modern portfolio theory, risk and return tradeoff follows a hyperbolic path

Figure 1: Traditional Risk-Return Tradeoff

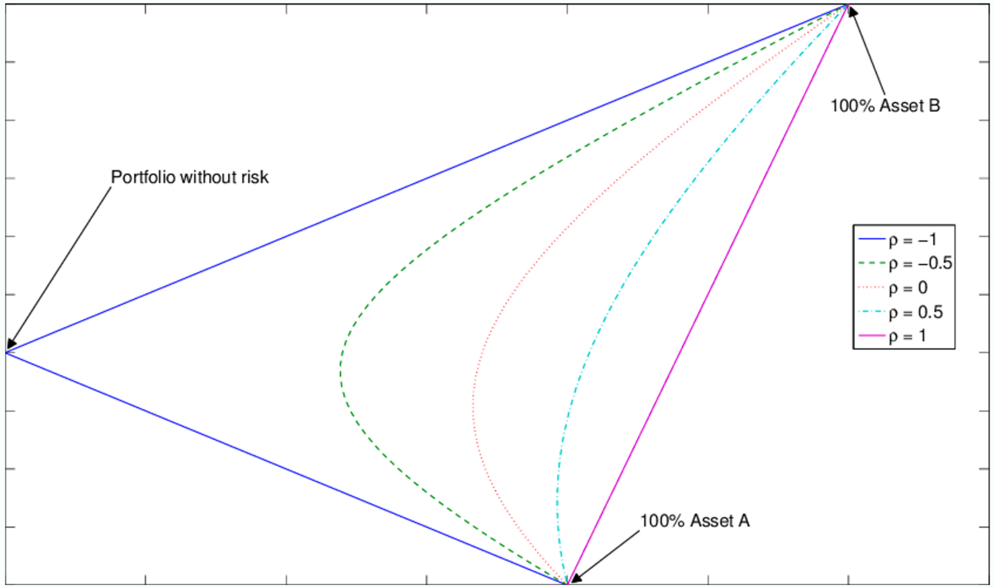
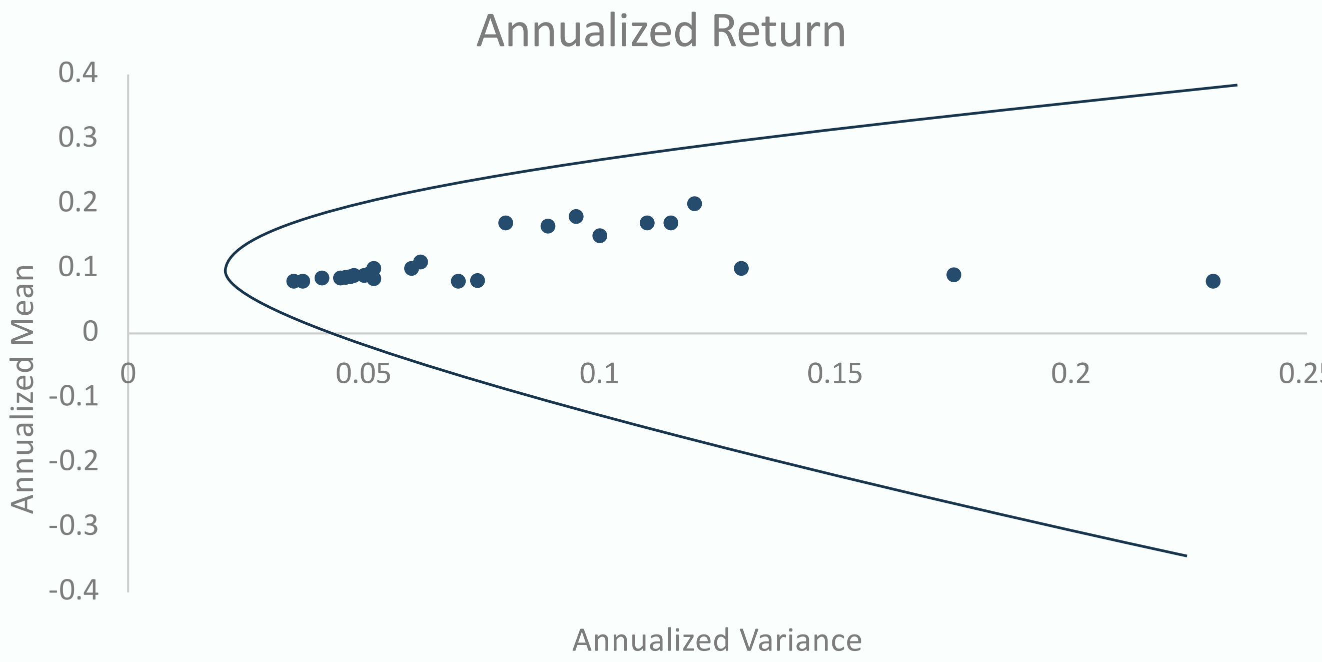


Figure 2: MPT Risk-Return Tradeoff

Diversification Across N Assets

With n securities, there is further potential for diversification

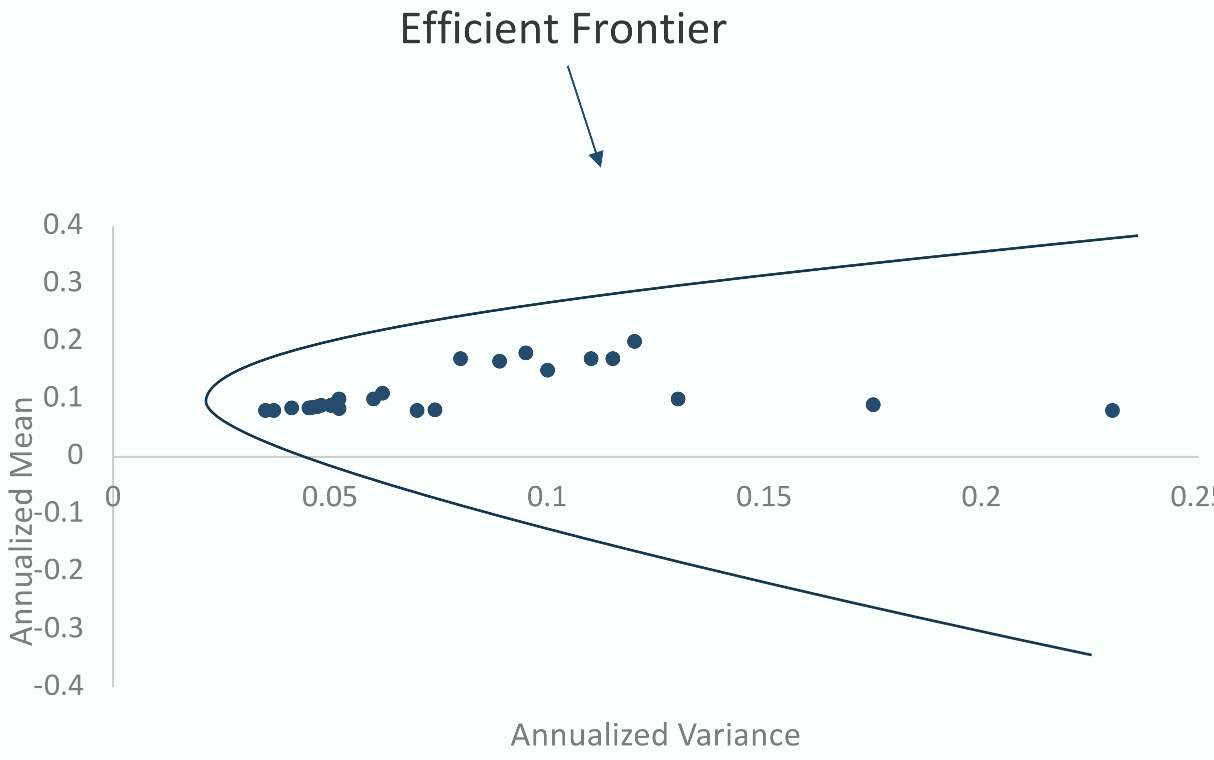
- A portfolio of $n > 1$ assets can be adjusted to result in different in-sample risk-return characteristics
- The set of all possible portfolios formed from this basis of assets forms a convex set in mean-variance space.
- The boundary of this set is known as the mean-variance frontier, and it forms a hyperbola.



Efficient Portfolios

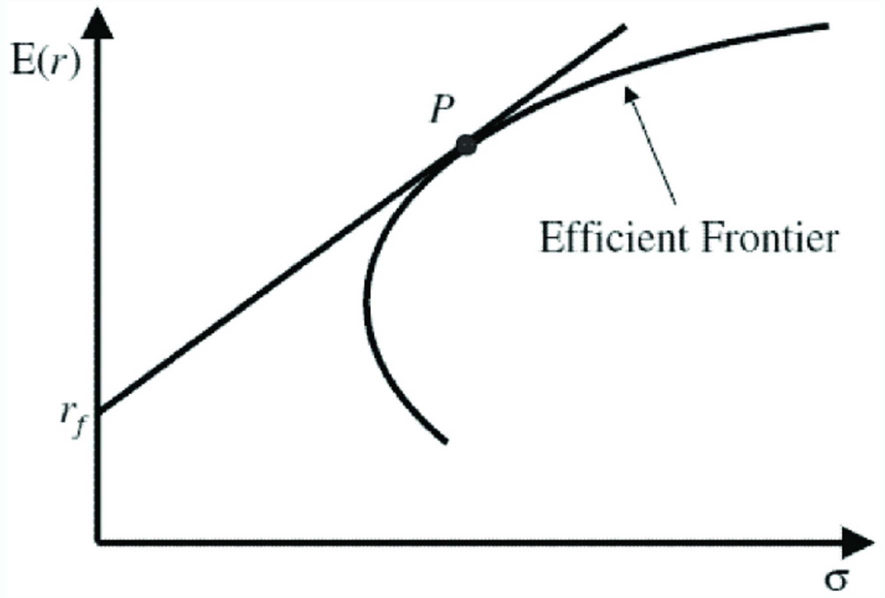
The top segment of the MV frontier is the set of efficient MV portfolios.

- Portfolios on this frontier have the maximum return at a given variance
  - You want to weight assets in your portfolio to have an overall portfolio on this efficient frontier
- Contrast this with the lower segment of the MV frontier, the inefficient MV portfolios.
  - The inefficient MV portfolios minimize mean return given the return variance.



Tangency Portfolio

- Assuming there exists a risk-free rate, there exists a portfolio on the efficient frontier that optimizes the in-sample portfolio Sharpe ratio.
  - This portfolio is the point where the capital market line is tangent to the efficient frontier
  - Capital market line shows risk-return tradeoff for MV investors
  - Slope of the Capital Market Line is the maximal Sharpe ratio which can be achieved by any portfolio



- This portfolio is called the tangency portfolio and assumes you invest 100% of the portfolio into risky assets.

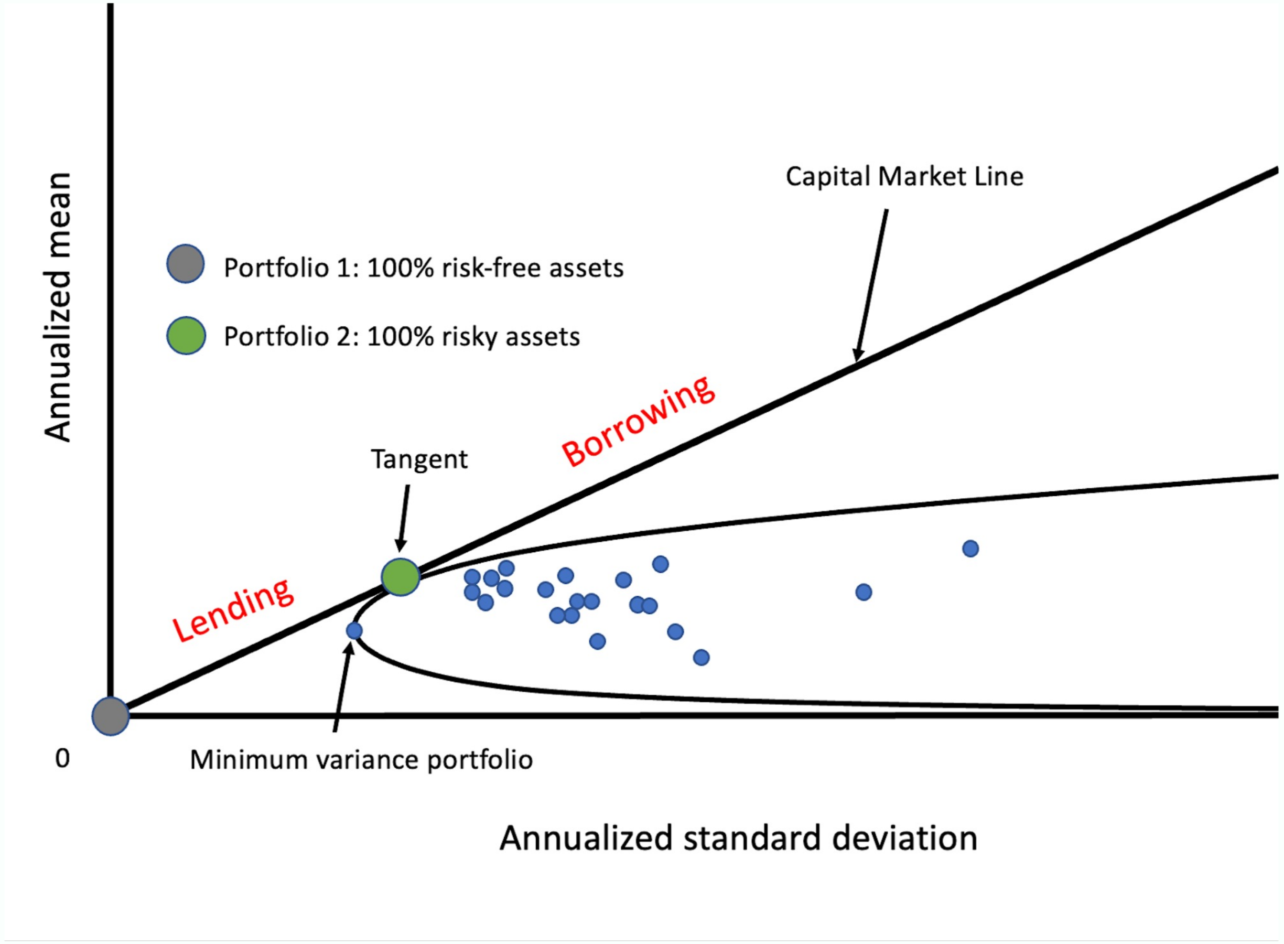
Two-Fund Separation

- Every Mean-Variance portfolio is the combination of the risky portfolio with the maximal Sharpe Ratio and the risk-free rate
- Thus, for a Mean-Variance investor, the asset allocation decision can be broken into two parts:
  - Find the tangency portfolio of risky assets, w_t
  - Choose an allocation between the risk-free rate and the tangency portfolio

Intuition of Asset Allocation

- The two-fund separation says that:
  - Any investment in risky assets should be in the tangency portfolio since it offers the maximum Sharpe Ratio.
  - One must decide the desired level of risk in the investment, which determines the split between the riskless asset and the tangency portfolio.

Note: Lending portion assumes a combination of the risk-free asset and the risky assets to form the portfolio. You're "lending" to the provider of the risk-free asset by incorporating it into your portfolio. Borrowing is the opposite; you're borrowing money from a riskless lender to invest more into risky assets to gain higher return through leverage but also higher risk



Notation

Suppose there are n risky assets

- r is an $n \times 1$ random vector. Each element is the return on one of the n assets.
- Let μ denote the $n \times 1$ vector of mean returns. Let Σ denote the $n \times n$ covariance matrix of returns.

$$\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{r}]$$

$$\Sigma = \mathbb{E}\left[(\boldsymbol{r} - \boldsymbol{\mu})(\boldsymbol{r} - \boldsymbol{\mu})'\right]$$

$$\Sigma = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{cccc} A & B & C & D \\ \end{array}$$

$$\Sigma = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0.3 & 0.2 & 0.1 & 0.6 \\ B & 0.4 & 0.5 & 0.2 & 0.8 \\ C & 0.2 & 0.8 & 0.5 & 0.9 \\ D & 0.7 & 0.3 & 0.3 & 0.4 \end{array}$$

- Assume Σ is positive definite—no asset is a linear function of the others.

With a Riskless Asset

- There are n risky assets available, with returns as r
- An investor chooses a portfolio, defined as a $n \times 1$ vector of allocation weights, w, in those n risky assets
- Since the total portfolio allocations must add to one, we have allocation to the risk-free rate $= 1 - w'1$

Mean Excess Returns

- Let μp denote the mean return on a portfolio.

$$\mu^p = \left(1 - \mathbf{w}'\mathbf{1}\right) r^f + \mathbf{w}'\boldsymbol{\mu}$$

- Use the following notation for excess returns:

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - \mathbf{1} r^f$$

- Thus, the mean return and mean excess return of the portfolio are

$$\mu^p = r^f + \mathbf{w}'\tilde{\boldsymbol{\mu}}$$

$$\tilde{\mu}^p = \mathbf{w}'\tilde{\boldsymbol{\mu}}$$

- Let μ^p denote the mean return on a portfolio.

Variance of Return

- The risk-free rate has zero variance and zero correlation with any security.
- Let Σ continue to denote the n × n covariance matrix of risky assets
- The return variance of the portfolio, wp is

$$\sigma_p^2 = \mathbf{w}'\Sigma\mathbf{w}$$

Calculating Mean-Variance Portfolio Weightings

- w^t is a nx1 vector, where each value in the vector is the suggested weighting allocation to that individual risky asset.

$$\boldsymbol{w}^{t} = \underbrace{\left(\frac{1}{\mathbf{1}'\Sigma^{-1}\tilde{\boldsymbol{\mu}}}\right)}_{\text{scaling}} \Sigma^{-1}\tilde{\boldsymbol{\mu}}$$

Two-Fund Allocation Adjustment

- A portfolio on the capital market line can be created through a scalar operation onto the tangency portfolio.

$$\mathbf{w}^* = \tilde{\delta}\,\mathbf{w}^{\mathrm{t}}$$

- Where the scalar value is calculated using the following formula

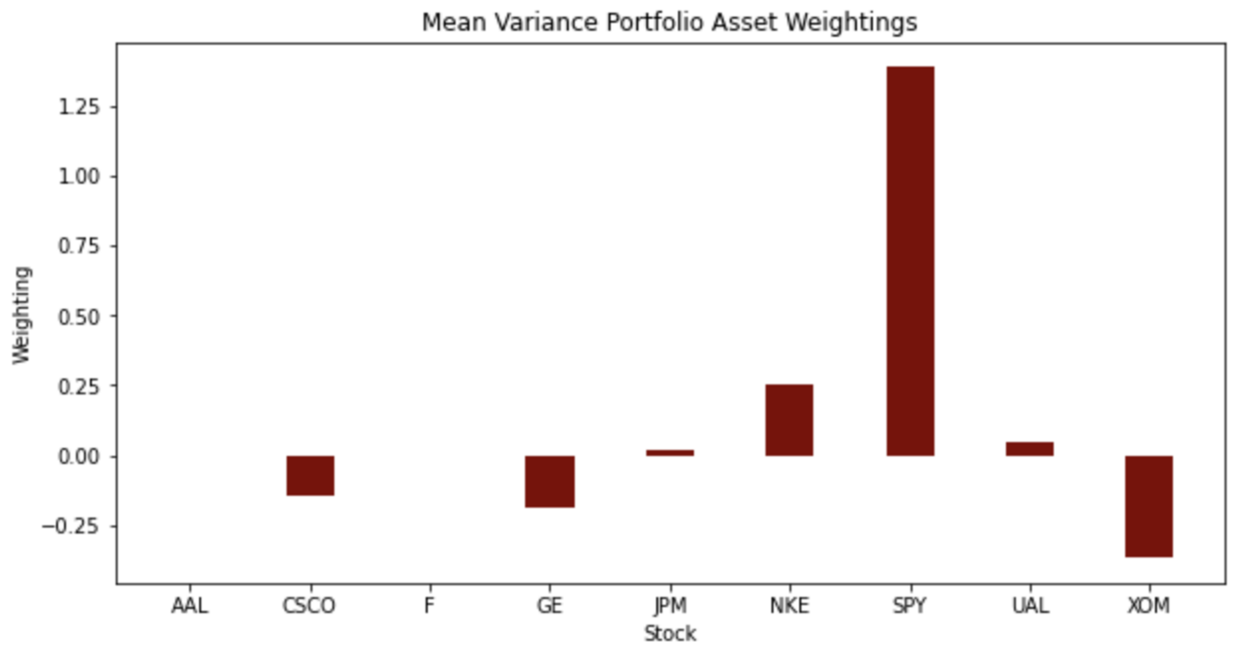
$$\tilde{\delta} = \left(\frac{\mathbf{1}'\Sigma^{-1}\tilde{\boldsymbol{\mu}}}{(\tilde{\boldsymbol{\mu}})'\Sigma^{-1}\tilde{\boldsymbol{\mu}}}\right)\tilde{\mu}^{p}$$

- The resulting weights give a portfolio that is on the capital market line and has an adjusted level of risk and return

Problems with Mean Variance

- Sensitized to In-Sample
- Market movements change and portfolio is not great out of sample
- Unrealistic weightings

AAL	-0.004421
CSCO	-0.146570
F	-0.000828
GE	-0.189041
JPM	0.018942
NKE	0.252251
SPY	1.390724
UAL	0.048748
XOM	-0.369806

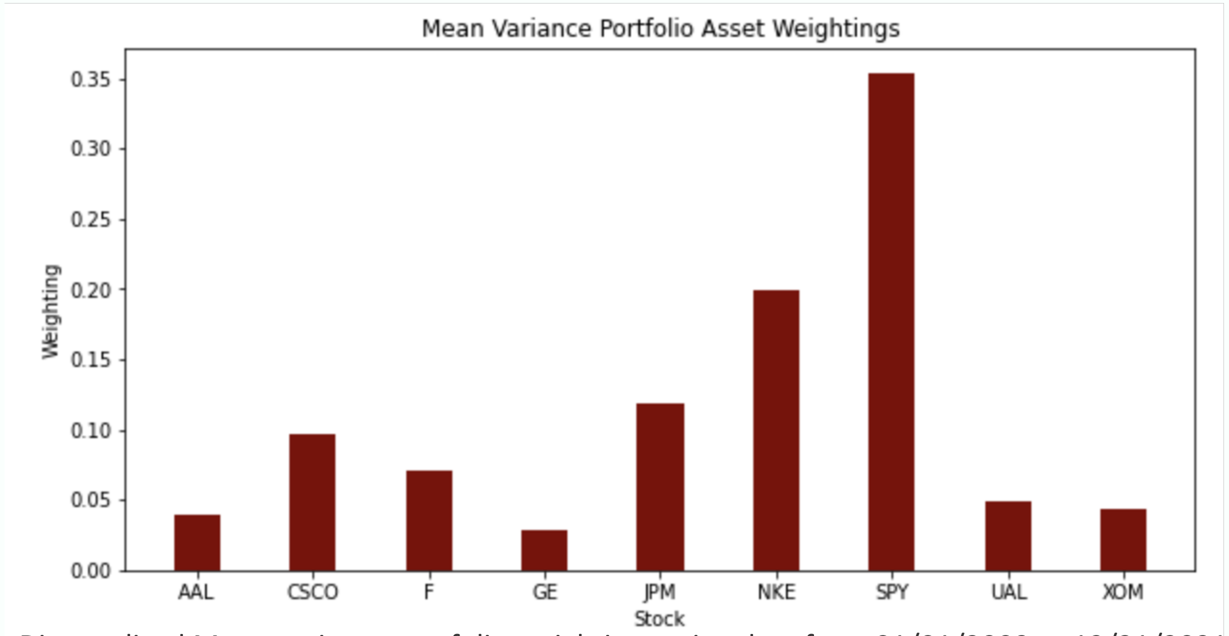


Mean-variance portfolio weightings using data from 01/01/2009 to 12/31/2021

Mean-Variance Alteration: Diagonalization

- Mean-variance relies on the pairwise covariance matrix as the measure of portfolio volatility
  - Using entire matrix makes it too sensitive to in-sample data
- You can diagonalize the covariance matrix to improve the allocation and performance out-of-sample
  - Only individual asset variances left (does not use covariance to other assets)
  - Less sensitive to in-sample data
  - No covariances to other assets means long-only weightings (formula can't use long-short paradigm to balance risk when there's no relational risk)

$$
\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}
\begin{array}{c}
\begin{array}{cccc} A & B & C & D \end{array} \\
\begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.6 \\ 0.4 & 0.5 & 0.2 & 0.8 \\ 0.2 & 0.8 & 0.5 & 0.9 \\ 0.7 & 0.3 & 0.3 & 0.4 \end{bmatrix}
\end{array}
\longrightarrow
\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}
\begin{array}{c}
\begin{array}{cccc} A & B & C & D \end{array} \\
\begin{bmatrix} 0.4 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.8 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}
\end{array}
$$



Diagonalized Mean-variance portfolio weightings using data from 01/01/2009 to 12/31/2021

Factor Decomposition

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Factor Decomposition

- You can break down an asset or portfolio's return onto "factors" using a regression
  - Factors are portfolios or attributes that are said to explain individual asset returns
    - The entire market portfolio is an example of a factor
- Use Cases:
  - Performance breakdown
  - Tracking
  - Hedging

Performance Breakdown

Factor models can be used as a measurement of fund performance

- Fund managers or portfolios can have impressive returns, but we want to know how the returns were generated
  - Was it due to strong correlation to a fast-growing market
  - Did the portfolio take on excessive risk

Linear Factor Decomposition

$$r = \alpha + \beta x + \varepsilon$$

- Decompose the returns and variation by running a regression of the portfolio (regressand) onto the factor (regressor)
  - Alpha is the return not explained by the factor
  - Beta and ε are the variation explained and unexplained by the benchmark, respectively
  - R^2 states how well the factor explains the variation of returns
    - Should have a high R^2
    - Low R^2 means you're not decomposing the returns onto the proper benchmark (for example, a technology fund should use the Nasdaq as the benchmark and not the S&P 500).

Understanding the Results

- Alpha
  - Sensitive to which benchmark you use
  - High alpha could mean strong performance or using an improper benchmark
    - Low R^2 meaning model is not capturing risk properly
- Could be missing beta from the model, meaning not all risk was captured

Performance Metrics

These metrics measure return-to-risk breakdown and are useful for comparing different assets and portfolios

Sharpe Ratio	Information Ratio	Treynor Ratio
Performance metric using variance as the measure of volatility	Measures the non-factor performance of the regressand	Performance metric using beta as the measure of volatility

$$\text{Sharpe Ratio} = \frac{\tilde{\mu}^p}{\sigma^p}$$

$$\text{IR} = \frac{\alpha}{\sigma_\epsilon}$$

$$\text{Treynor Ratio} = \frac{\mathbb{E}\left[\tilde{r}^i\right]}{\beta^{i,m}}$$

Hedging

Suppose someone wants to invest in asset r but wants to remove the risk related to the overall market movements

- Run the regression of the asset Y onto the market portfolio x
  - $Y = \alpha + \beta x + \varepsilon$
- For every $1.00 invested into Y, the fund can short-sell β*$1.00 of x.
- The fund is then holding
  - $Y - \beta x = \alpha + \varepsilon$
  - Where the remaining returns are those unexplained by the market

Tracking

You can construct a portfolio that tracks the returns of another asset

- Assume you don't know what assets are in a portfolio but you have its return data.
  - You have K assets that you want to invest in to track the above portfolio
- Run the multivariate regression where y is the portfolio you want to track
  - $y = \alpha + \beta^1 x^1 + \beta^2 x^2 + \ldots \beta^k x^k + \varepsilon$
- To track the portfolio, invest β^i dollars into each of the assets x^i
- R^2 measures how well your tracking portfolio replicates the original portfolio

Linear Factor Pricing Models

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Factor Pricing

Via no-arbitrage arguments (all securities of same type are priced the same),

- The expected (excess) return of an asset is explained by the tangency portfolio and its relation (beta) to that portfolio

$$E\left(\tilde{r}^i\right) = \beta^{i,t} E\left(\tilde{r}^t\right)$$

$$\beta^{i,t} \equiv \frac{\mathrm{cov}(\tilde{r}^i, \tilde{r}^t)}{\mathrm{var}(\tilde{r}^t)}$$

Linear Factor Pricing Models (LFPM)

LFPMs are assertions about the identity of the tangency portfolio

- Investors don't allocate to the Mean Variance Portfolio
- Assumes mean-variance portfolio is for pricing expected returns
- However, even if you don't want to hold a MV portfolio, all E(r)s are calculated as covariances to the MV portfolio.

Capital Asset Pricing Model

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The Market Portfolio

CAPM identifies the market portfolio as the tangency portfolio.

- Market portfolio is the value-weighted portfolio of all available assets
- In practice, a broad equity index like SPY is generally used

The CAPM

- The most famous of these linear factor models is the CAPM, or Capital Asset Pricing Model

$$E\left(\tilde{r}^{i}\right) = \beta^{i,m} E\left(\tilde{r}^{m}\right)$$

$$\beta^{i,m} \equiv \frac{\mathrm{cov}(\tilde{r}^{i}, \tilde{r}^{m})}{\mathrm{var}(\tilde{r}^{m})}$$

1. The r with a tilda on top denotes the excess return of an asset

Expected Returns

The CAPM tells you about expected returns:

- E(r) for any asset is a function of the risk-free rate and the market risk premium.
- Beta is determined using a regression. CAPM doesn’t tell you how the risk-free rate or market risk premium are given.

The Role of Beta

- The CAPM says that the market beta is the **only** risk associated with higher average returns.
  - Because you can diversify, investors don’t charge higher for non-correlated (idiosyncratic) risk.
  - Higher systematic risk demands higher return

Expected vs Realized Returns

- The CAPM implies that expected returns for any security are

$$E\left(\tilde{r}^{i}\right) = \beta^{i,m} E\left(\tilde{r}^{m}\right)$$

- which implies that realized returns can be written as

$$\tilde{r}_t^{i} = \beta^{i,m} \tilde{r}_t^{m} + \epsilon_t$$

- The CAPM implies that αi = 0 for every single αi because the market tangency portfolio should perfectly predict all asset returns and no returns should be unexplained by the market

Criticism of the CAPM

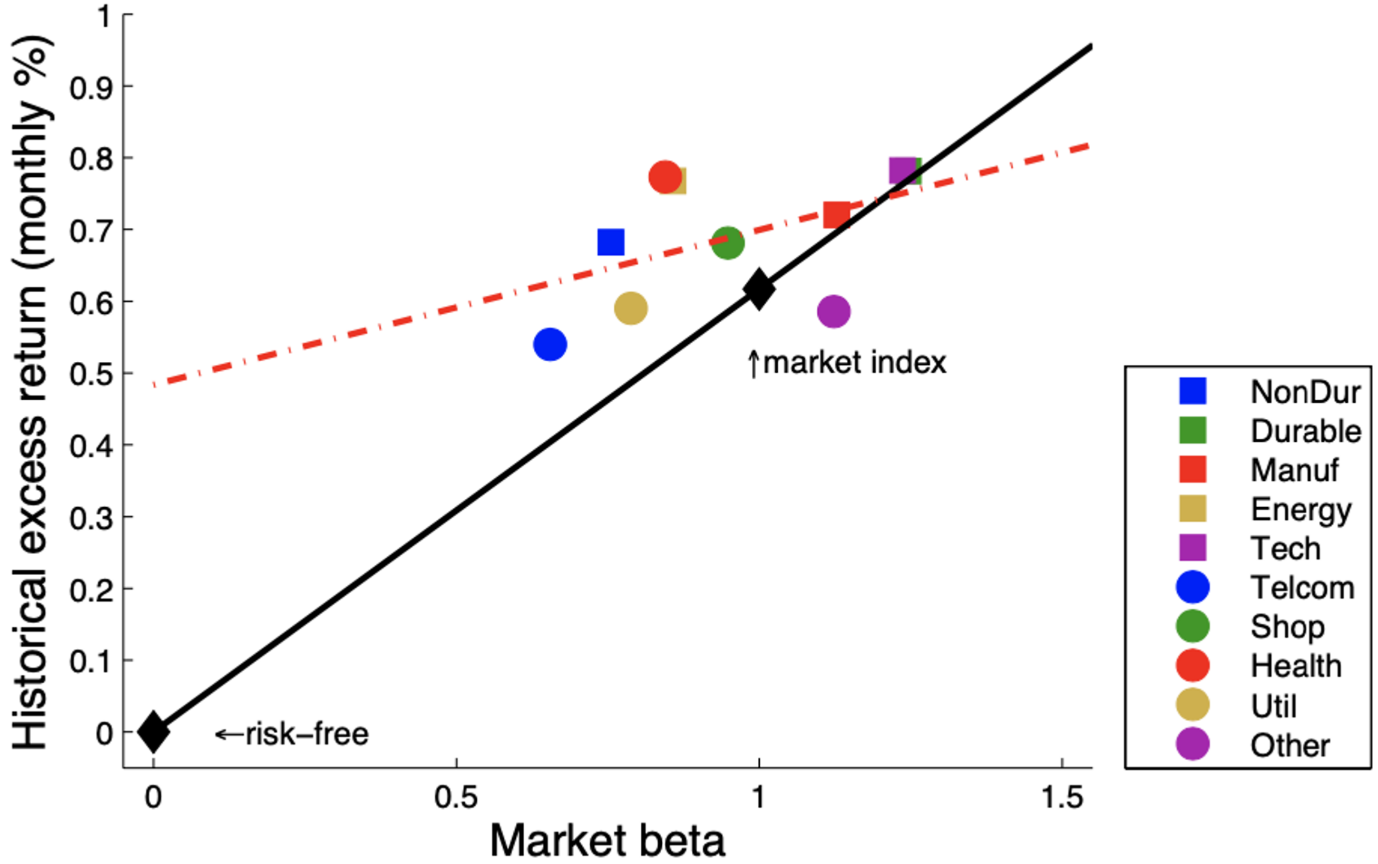

Figure: Data Source: Ken French. Monthly 1926-2011.

The risk-reward tradeoff is too flat relative to CAPM.

Criticisms of the CAPM and LFPM

The risk return tradeoff shown below is too flat relative to the CAPM

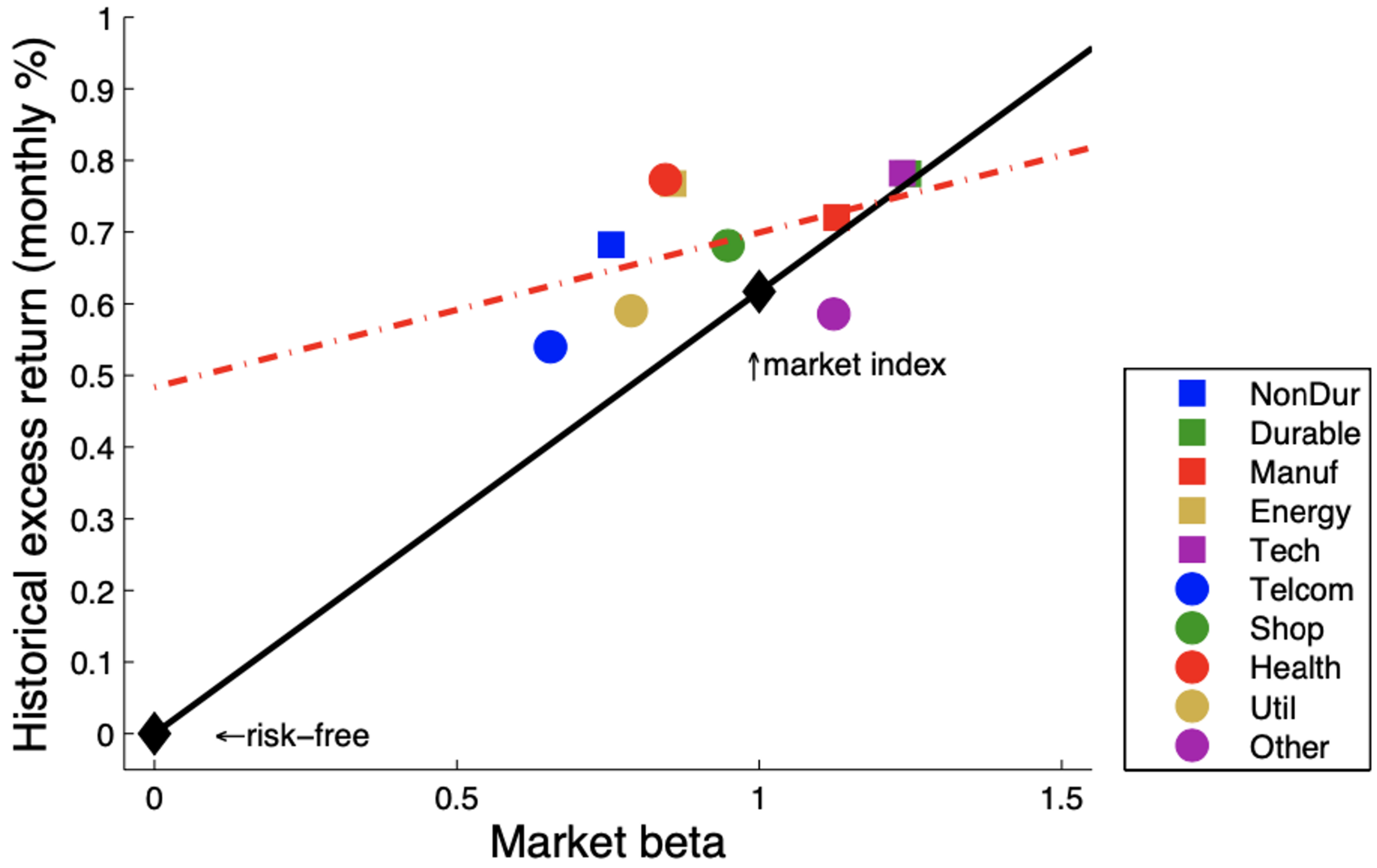

Figure: Data Source: Ken French. Monthly 1926-2011.

Multi-Factor Models

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Multiple Factors

Multifactor models price asset returns by assuming the tangency portfolio is a linear combination of multiple factors (these factors don't have to sum to one)

- Expected return can be calculated by the asset's total relation to each of the factors
- Suppose we have an array of factor returns such that the tangency portfolio is a linear combination of them

$$\tilde{\boldsymbol{r}}^{\boldsymbol{t}} = \boldsymbol{w}'\tilde{\boldsymbol{r}}^{\boldsymbol{z}}$$

$$\boldsymbol{E}(\tilde{\boldsymbol{r}}^{\boldsymbol{p}}) = (\boldsymbol{\beta}^{\boldsymbol{p},\boldsymbol{z}})'\boldsymbol{E}(\tilde{\boldsymbol{r}}^{\boldsymbol{z}})$$

Fama-French Model

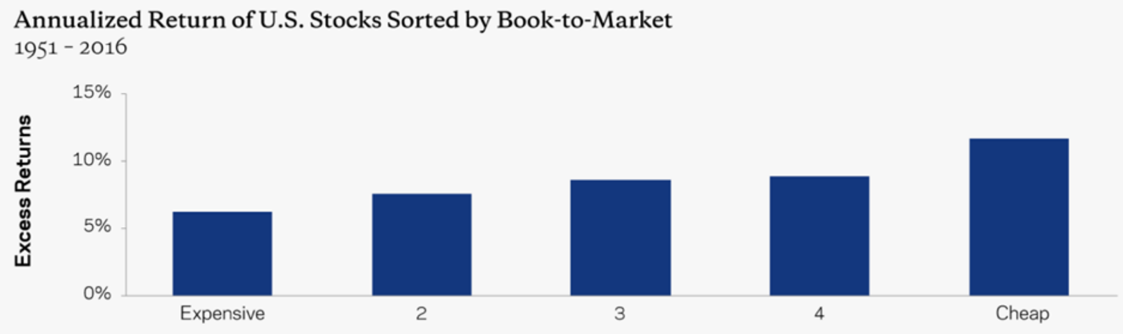
- The Fama-French 3-factor model is one of the best-known multifactor models

$$E(\tilde{r}^i) = \beta^{i,m} E(\tilde{r}^m) + \beta^{i,s} E(\tilde{r}^s) + \beta^{i,v} E(\tilde{r}^v)$$

- Here,
  - r^m Is the excess market return like in the CAPM
  - r^s is a portfolio that goes long small stocks and shorts large stocks
  - r^v is a portfolio that goes long value stocks and shorts growth stocks

Fama-French Measure of Value

- Historically, value stocks have delivered higher average returns.
  - Measured by metrics such as P / BV or EV / EBITDA
- High B/M stocks are **value stocks**
- Low B/M stocks are **growth stocks**



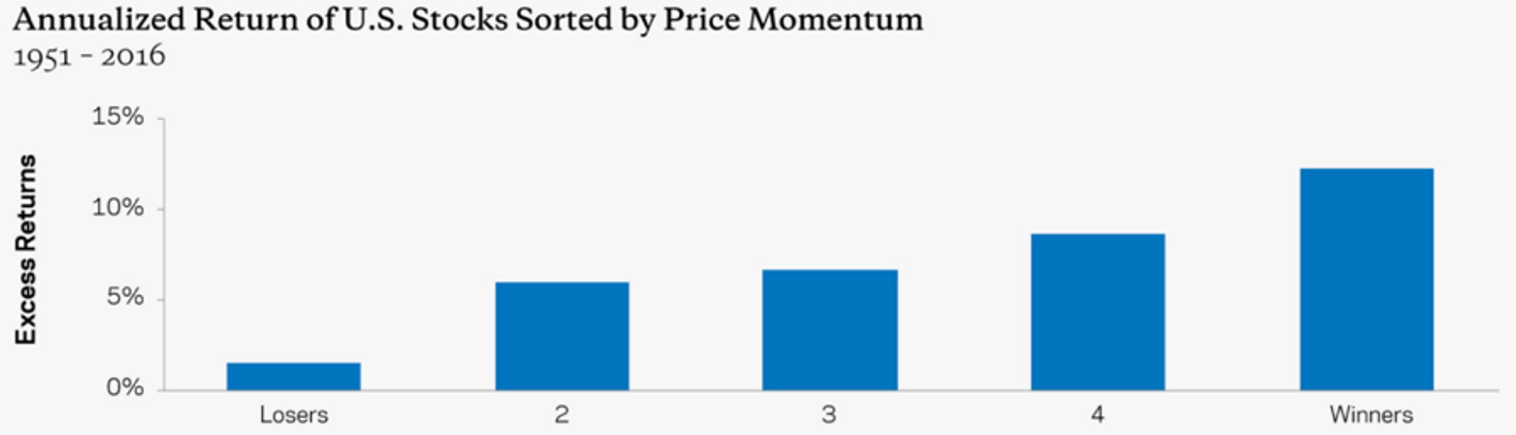
Sources: AQR and Kenneth R. French Data Library. Portfolios from Kenneth R. French Data Library formed based on book-to-market; quintiles are equal-weighted; returns are excess of cash. Returns sourced from "Portfolios Formed on Book-to-Market." See Kenneth R. French Data Library for further details. These are not the returns of an actual portfolio AQR manages and are for illustrative purposes only. Past performance is not a guarantee of future performance.

Other Factors

There are many other factors that “claim” to explain asset returns

- Investments funds, like AQR and Blackrock, invest heavily into finding the perfect factors to predict asset returns

- Examples include
  - Momentum
  - Inflation
  - Size



Source: AQR and Kenneth R. French Data Library. Portfolios from Kenneth R. French Data Library formed based on 12-month momentum, skipping most recent month; quintiles are equal-weighted; returns are excess of cash. Returns sourced from “10 Portfolios Formed on Momentum.” See Kenneth R. French Data Library for further details. These are not the returns of an actual portfolio AQR manages and are for illustrative purposes only. Past performance is not a guarantee of future performance.

Factor Investing in Practice

You can develop portfolios based off factor analysis

Smart Beta

- Broad encompassing term
- Traditional ETFs weight by market cap
  - Automatic adjustments
  - Favors overpriced assets
- Smart Beta applies a "factor tilt" to adjust weightings
- Example (There are many variations):
  - Sxi -> Cross-sectional Z-Score
  - Wi -> Traditional weighting scheme (i.e market cap weighted)
  - U -> Entire universe of assets

Systematic Factor Portfolio Construction

- Runs regression of all equities onto factor portfolios
- Applies parameters to results to create portfolio
- Systematically implemented

Constructing Multi-Factor Portfolios

Heuristic Construction	Optimized Construction

Heuristic Construction

- Heuristic construction creates a total factor score based off relation to given factors
- Example construction (where $F_{i,j}$ is the factor exposure between asset I and factor j
- Ranks the assets based off the score (i.e top 500)
- Portfolio is constructed based off rankings

$$\alpha_i = 0.2 * F_{1,i} + 0.2 * F_{2,i} + 0.2 * F_{3,i} + 0.2 * F_{4,i} + 0.2 * F_{5,i}$$

Optimized Construction

- Creates a risk framework for large scale optimization
- Apply various security and sector level constraints
- Where, Σ is the security covariance matrix and σ_p is the expected portfolio tracking error that aims to minimize.

$$\min_{W} \bar{\sigma}_p = W^T \Sigma W$$

Subject to

$$0 \le w_i \le j \text{ for every stock } i$$

$$sum(w_i * f_{k,i}) = F_k \text{ for every factor } k$$

$$sum(w_i * s_{l,i}) = S_l \text{ for every sector } l$$

$$sum(w_i) = 1$$

Quantitative Portfolio Management in Practice

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See Associated Code for Application

https://colab.research.google.com/drive/1DH1Y_8PCu7bTRz6EONqb8Nq4GAqAj2Ae?authuser=1